Variants of Ahlfors' lemma and properties of the logarithmic potentials *

Tanran Zhang

Abstract

As a special class of conformal metrics with negative curvatures, SK-metrics play a crucial role in metric spaces. This paper concerns the variants of Ahlfors' lemma in an SK-metric space and gives a higher order derivative formula for the logarithmic potential function, which can be applied for the estimates near the singularity of a conformal metric with negative curvatures.

1 Introduction

Let $\mathbb{C}$ be the complex plane, $\mathbb{D}$ be the open unit disk in $\mathbb{C}$ and $\mathbb{D}^* := \mathbb{D} \setminus \{0\}$ be the punctured unit disk. Let $\mathcal{S}$ denote a Riemann surface. Let A be the basic family consisting of homeomorphisms σ defined on the plane domains of $\mathbb{C}$ into $\mathcal{S}$ which defines the conformal structure of $\mathcal{S}$. In our discussion we take the linear notation for a conformal metric $ds = \lambda(z)|dz|$ and set the density function $\lambda(z)$ to be positive on $\mathcal{S}$. However, in Heins' paper [4], he let the function $\lambda(z)$ be non-negative for a conformal metric $\lambda(z)|dz|$ on $\mathcal{S}$, and then defined the SK-metrics.

Let $\mathbb{P}$ denote the Riemann sphere $\mathbb{C} \cup \{\infty\}$ with its canonical complex structure and a subdomain $\widetilde{\Omega} \subset \mathbb{P}$. For a point $p \in \widetilde{\Omega}$, let z be the local coordinates such that $z(p) = 0$. We say that a conformal metric $\lambda(z)|dz|$ on the punctured domain $\Omega := \widetilde{\Omega}\backslash\{p\}$ has a *singularity of order* $\alpha \leq 1$ *at the point* p, if, in local coordinates z,

$$\log \lambda(z) = \begin{cases} -\alpha \log |z| + O(1) & \text{if } \alpha < 1 \\ -\log |z| - \log \log(1/|z|) + O(1) & \text{if } \alpha = 1 \end{cases} \tag{1.1}$$

as $z \to 0$, with O being the Landau symbol throughout our study. Let $M_u(r) := \sup_{|z|=r} u(z)$ for a real-valued function $u(z)$ defined in a punctured neighborhood of $z = 0$ and call

$$\alpha(u) := \lim_{r\to 0^+} \frac{M_u(r)}{\log(1/r)} \tag{1.2}$$

the *order of* $u(z)$ if this limit exists. For $u(z) := \log \lambda(z)$, $\alpha(u)$ in (1.1) and (1.2) are equivalent. We call the point p a *conical singularity* or *corner of order* α if $\alpha < 1$ and a *cusp* if $\alpha = 1$. The *generalized Gaussian curvature* $\kappa_\lambda(z)$ of the density function $\lambda(z)$ is defined by

$$\kappa_\lambda(z) = -\frac{1}{\lambda(z)^2} \liminf_{r\to 0} \frac{4}{r^2}\left(\frac{1}{2\pi}\int_0^{2\pi} \log \lambda(z + re^{it})dt - \log \lambda(z)\right). \tag{1.3}$$

*Keywords. Gaussian curvatures, SK-metrics, logarithmic potentials.

We say a conformal metric $\lambda(z)|dz|$ on a domain $\Omega \subseteq \mathbb{C}$ is *regular*, if its density $\lambda(z)$ is of class C^2 in Ω. We will show that, if $\lambda(z)|dz|$ is a regular conformal metric, then (1.3) is equivalent to

$$\kappa_\lambda(z) = -\frac{\Delta \log \lambda(z)}{\lambda(z)^2}, \tag{1.4}$$

where Δ denotes the Laplace operator. We will discuss details in Lemma 2.3. It is well known that, if $a < \kappa_\lambda(z) < b < 0$, the metric $\lambda(z)|dz|$ only has corners or cusps at isolated singularities, see [8]. If $\kappa_\lambda(z) \geq 0$ and the energy is finite, then only corners occur, see [5, 10], also [7].

We have a fact that the Gaussian curvature is a conformal invariant. Let $\lambda(z)|dz|$ be a conformal metric on a plane domain D and $f : W \to D$ be a holomorphic mapping of a Riemann surface W into D. Then the metric $ds = f^*\lambda(w)|dw|$ on W induced by f from the original metric $\lambda(z)|dz|$ is called the *pullback of* $\lambda(z)|dz|$ and defined by

$$ds = f^*\lambda(w)|dw| := \lambda(f(w))|f'(w)||dw|. \tag{1.5}$$

It is evident that $f^*\lambda(w)|dw|$ is a conformal metric on $W\backslash\{\text{critical points of } f\}$ with Gaussian curvature

$$\kappa_{f^*\lambda}(w) = \kappa_\lambda(f(w)).$$

Using this conformal invariance, we can only consider one Riemann surface with the conformal metric all over its conformal equivalence class.

The paper is divided into two parts. Section 2 is devoted to a class of conformal metrics, SK-metrics. We begin with the definition and then discuss the maximum principle for these metrics. Section 3 is about the potential theory, which is a main tool we use to study the local term of a conformal metric near the singularity.

2 SK-metrics

For a topological space X, a function $u : X \to [-\infty, \infty)$ is called *upper semi-continuous* if the set $\{z \in X : u(z) < \alpha\}$ is open in X for each real number α. If φ is a *uniformizer* of $\mathcal{S}$, i.e. φ is a univalent conformal map of a plane domain into $\mathcal{S}$, then the conformal density function λ on $\mathcal{S}$ can be extended to $\mathcal{S} \cup \{\varphi\}$ such that the extension is a conformal metric relative to $\mathcal{S} \cup \{\varphi\}$. We denote the image of φ with respect to this extension by λ_φ and call it the *φ-scale of* λ. For two uniformizers φ and ψ of $\mathcal{S}$, $\varphi(a) = p$, $\psi(b) = p$, $a, b \in \mathcal{S}$, $p \in \mathbb{C}$, if λ_φ and λ_ψ are upper semi-continuous at a and b respectively, we say λ is upper semi-continuous at p. If λ is upper semi-continuous at every point of $\mathcal{S}$, we say λ is upper semi-continuous on $\mathcal{S}$.

Now we give the definition for SK-metrics. The concept of SK-metrics was given by Heins in [4], but its initial idea came from Ahlfors in [1] and [2]. Heins defined the SK-metrics by means of the Gaussian curvature and proved a more general maximum principle as a variant of Ahlfors' lemma. According to Heins, the terminology of "SK-metric" is partly because its (Gaussian) "curvature is subordinate to -4", see [4, p.3]. We call an upper semi-continuous metric $\lambda(z)|dz|$ on $\mathcal{S}$ an *SK-metric* if its Gaussian curvature is bounded above by -4 at those points z in $\mathcal{S}$ where z satisfies $\lambda(z) > 0$. A complete

metric with the negative constant Gaussian curvature is called the hyperbolic metric. The hyperbolic metric on the unit disk $\mathbb{D}$ is defined by

$$d\sigma = \lambda_{\mathbb{D}}(z)|dz| = \frac{|dz|}{1-|z|^2} \tag{2.1}$$

with the constant Gaussian curvature -4 and it is an extremal SK-metric. For SK-metrics, there is a generalization of the maximum principle mentioned by Ahlfors [2, Theorem A] and Heins [4, Theorem 2.1], which claims that the hyperbolic metric is the unique maximal SK-metric on $\mathbb{D}$.

Theorem 2.1 [2] (Ahlfors′ lemma) *Let $d\sigma$ be the hyperbolic metric on $\mathbb{D}$ given in (2.1) and ds be the metrics on $\mathbb{D}$ induce by an SK-metric on some Riemann surface W. If the function $f(z)$ is analytic in $\mathbb{D}$, then the inequality*

$$ds \leq d\sigma$$

will hold throughout the circle.

The following result stated on the corresponding Riemann surface. It is a variant of Theorem 2.1.

Theorem 2.2 [4] *Suppose that W is a relatively compact domain of $\mathcal{S}$ and that $\lambda(w)$ is an SK-metric on W, $\mu(w)$ is a pullback on W of $\lambda_{\mathbb{D}}(z)$ defined in (2.1). If for all $p \in \partial W$,*

$$\limsup_{w\to p} \frac{\lambda(w)}{\mu(w)} \leq 1,$$

then throughout the boundary ∂W, it holds that

$$\lambda(w) \leq \mu(w).$$

Heins used condition (1.3) to define SK-metrics and he mentioned the equivalence between (1.3) and (1.4) for SK-metrics in one word, see [4, (1.4)]. Here we present it in detail.

Lemma 2.3 *Suppose $\Omega \in \mathbb{C}$ is a domain. If a function u is of class $C^2(\Omega)$, then for $z \in \Omega$, we have*

$$\lim_{r\to 0} \frac{4}{r^2}\left(\frac{1}{2\pi}\int_0^{2\pi} u(z+re^{it})dt - u(z)\right) = \Delta u(z).$$

Proof. Since u is of C^2, using Taylor's expansion of $u(z)$ at $z_0 \in \Omega$,

$$u(z_0+z) = u(z_0)+u_x(z_0)\cdot x+u_y(z_0)\cdot y+\frac{1}{2}u_{xx}(z_0)\cdot x^2+\frac{1}{2}u_{yy}(z_0)\cdot y^2+u_{xy}(z_0)\cdot xy+\varepsilon(z),$$

where $\varepsilon(z) \to 0$ as $z \to 0$ and $z = x+yi$, we have

$$\frac{1}{2\pi}\int_0^{2\pi} u(z_0+re^{it})dt - u(z_0) = \frac{r^2}{4}\left(u_{xx}(z_0)+u_{yy}(z_0)\right) + \frac{1}{2\pi}\int_0^{2\pi} \varepsilon(z)dt.$$

As $r = |z| \to 0$,

$$\frac{1}{r^2}\int_0^{2\pi} \varepsilon(z)dt = \int_0^{2\pi} \frac{\varepsilon(z)}{r^2}dt = 0,$$

then

$$\lim_{r\to 0}\frac{4}{r^2}\left(\frac{1}{2\pi}\int_0^{2\pi} u(z+re^{it})dt - u(z)\right) = u_{xx}(z_0)+u_{yy}(z_0) = \Delta u(z_0)$$

as required. □

The following theorem offers us a simple way to construct a new SK-metric on a plane domain by means of the maximum principle. This is called the "gluing lemma".

Theorem 2.4 [7] *Let $\lambda(z)|dz|$ be an SK-metric on a domain $G \in \mathbb{C}$ and let $\mu(z)|dz|$ be an SK-metric on a subdomain U of G such that the "gluing condition"*

$$\limsup_{U\ni z\to\xi} \mu(z) \le \lambda(\xi)$$

holds for all $\xi \in \partial U \cap G$. Then $\sigma(z)|dz|$ defined by

$$\sigma(z) := \begin{cases} \max\{\lambda(z),\mu(z)\} & \text{for } \ z\in U, \\ \lambda(z) & \text{for } \ z \in G\setminus U \end{cases}$$

is an SK-metric on G.

We end this section with the discussion on SK-metrics in the punctured domain. On the punctured unit disk $\mathbb{D}^*$, the hyperbolic metric is expressed by

$$\lambda_{\mathbb{D}^*}(z)|dz| = \frac{|dz|}{2|z|\log(1/|z|)} \tag{2.2}$$

with the constant curvature -4. From the definition (1.1) of the singularity and its order, we know that the metric (2.2) has order 1 at the origin. In any punctured disk, Kraus, Roth and Sugawa gave the expression of the hyperbolic metric which has a singularity at the origin of order $\alpha < 1$ in [7] without any detailed discussion. Now we give a complete presentation of the proof.

Theorem 2.5 [7] *For $R > 0$, let $D_R^* := \{z : 0 < |z| < R\}$ and*

$$\lambda_{\alpha,R}(z) := \begin{cases} \dfrac{1-\alpha}{2|z|\sinh\left((1-\alpha)\log\frac{R}{|z|}\right)} & \text{if } \ \alpha < 1, \\ \dfrac{1}{2|z|\log\frac{R}{|z|}} & \text{if } \ \alpha = 1 \end{cases}$$

for $z \in D_R^*$, then for an arbitrary SK-metric $\sigma(z)$ on D_R^* which has a singularity at $z = 0$ of order α, we have $\sigma(z) \le \lambda_{\alpha,R}(z)$.

Proof. We consider the case $\alpha < 1$. First, choose an arbitrary $0 < R_0 < R$, consider $\lambda_{\beta,R_0}(z)$ on $0 < |z| < R_0$ for $\alpha < \beta < 1$, and let $u(z) := \log\sigma(z)$, $v(z) := \log\lambda_{\beta,R_0}(z)$, $E := \{z : 0 < |z| < R_0, u(z) > v(z)\}$.

Now we have the assertion that $0 \notin \overline{E}$. Since $\sigma(z)|dz|$ and $\lambda_{\beta,R_0}(z)|dz|$ both have a singularity at $z = 0$ with order α, β respectively, then

$$v(z) = -\beta\log|z| + O(1),\ \ u(z) = -\alpha\log|z| + O(1),$$

so $u - v = (\beta-\alpha)\log|z| + O(1)$. Since $u - v \to -\infty$ as $z \to 0$, then on a sufficiently small neighborhood of $z = 0$, $u - v < 0$ holds, thus $0 \notin \overline{E}$.

Similarly, we have $\partial E \cap \{z : 0 < |z| < R_0\} = \emptyset$, because $v \to +\infty$ as $|z| \to R_0$, and u is bounded in $\{z : 0 < |z| < R_0\}$.

Then consider the curve $|z| = R_0$. It is clear that $v(z)$ and $u(z)$ satisfy

$$\lim_{r\to 0} \frac{4}{r^2} \left(\frac{1}{2\pi} \int_0^{2\pi} v(z + re^{it})dt - v(z) \right) = e^{2v},$$

and

$$\liminf_{r\to 0} \frac{4}{r^2} \left(\frac{1}{2\pi} \int_0^{2\pi} u(z + re^{it})dt - u(z) \right) \geq e^{2u}$$

by Lemma 2.3. Thus

$$\liminf_{r\to 0} \frac{4}{r^2} \left(\frac{1}{2\pi} \int_0^{2\pi} \left(u(z + re^{it}) - v(z + re^{it})\right) dt - (u(z) - v(z)) \right) \geq e^{2u} - e^{2v},$$

which is positive on E. By definition of limit inferior, we have for $z \in E$

$$\frac{1}{2\pi} \int_0^{2\pi} \left(u(z + re^{it}) - v(z + re^{it})\right) dt - (u(z) - v(z)) > 0,$$

therefore,

$$u(z) - v(z) \leq \frac{1}{2\pi} \int_0^{2\pi} \left(u(z + re^{it}) - v(z + re^{it})\right) dt.$$

Now we recall the definition of subharmonic functions. Let Ω be an open subset of $\mathbb{C}$. A function $u : \Omega \to [-\infty, \infty)$ is called subharmonic if u is upper semi-continuous and satisfies the local sub-mean inequality, i.e. given $z \in \Omega$, there exists $\rho > 0$ such that

$$u(z) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{it})dt \tag{2.3}$$

for $0 \leq r < \rho$. If we adopt the definition as above, then $u - v$ is subharmonic on E, hence $u - v$ has no maximum in E and $u - v$ approaches its least upper bound on a sequence tending to ∂E. A contradiction. So $E = \emptyset$.

Finally, letting $R_0 \to R$ and $\beta \to \alpha$ gives the maximality of $\lambda_{\alpha,R}(z)$ for $\alpha < 1$. According to Kraus, Roth and Sugawa, for the case $\alpha = 1$ this expression has to be interpreted in the limit sense $\alpha \nearrow 1$ to obtain $\lambda_{1,R}(z)$, i.e.

$$\lambda_{1,R}(z) = \lim_{\alpha \nearrow 1} \lambda_{\alpha,R}(z) = \frac{1}{2|z| \log \frac{R}{|z|}}$$

□

Remark. The righthand side of (2.3) is called the circumferential mean of function u. Heins used it to describe the curvature in the definition of SK-metrics in [4] with $\rho = 1$ and $z = 0$.

3 Potential theory

Generally speaking, the SK-metric is defined by the fact that its Gaussian curvature no greater than some negative constant. So the maximum principle for the SK-metric is common and useful. After a combination with PDEs, the asymptotic behavior of a metric has something in-between with the local properties of the solution to the corresponding PDE. We can consider the curvature equation

$$\Delta u = -\kappa(z)e^{2u}, \tag{3.1}$$

where $\kappa(z)$ is known, then the definition of SK-metrics is fruitful in the case that the curvature function $\kappa(z)$ is strictly negative and Hölder continuous in $\mathbb{D}^*$, see [6], also [9] for details. For an SK-metric $\lambda(z)$ on $\mathbb{D}^*$, regarding $u := \log \lambda$ as a solution to the equation (3.1), the global properties of u have been well known by means of the study on equation (3.1). However, near the singularity $z = 0$, the local properties are still not explicit. We can employ a way related to partial differential equations to investigate the asymptotic behavior of u near the origin. Potential theory is a powerful tool in our case. In this section, we present a formula of the higher order derivatives for the logarithmic potential, and give an asymptotic estimate for u near the origin without any proof as an application of potential theory. We only refer to the logarithmic potential.

We identify $\mathbb{C}$ with $\mathbb{R}^2$, and write $z = x_1 + ix_2$, $\zeta = y_1 + iy_2$. Set $0 < r \leq 1$ and denote $D_R := \{z \in \mathbb{C} : |z| < R\}$, ${D_R}^* := D_R \backslash \{0\}$ for a positive number R. For a bounded, integrable function $f(z)$ defined on a domain $\Omega \subseteq \mathbb{C}$, the integral

$$\frac{1}{2\pi} \iint_{\Omega} \log|z - \zeta| f(\zeta) d\sigma_\zeta$$

is called the *logarithmic potential of* f, where $d\sigma_\zeta$ is the area element. The Hölder spaces $C^{n,\nu}(D_R)$ are defined as the subspaces of $C^n(D_R)$ consisting of functions whose $n-$th order partial derivatives are locally Hölder continuous with exponent ν in D_R, $0 < \nu \leq 1$. Then the following proposition for the first and the second order derivatives of the logarithmic potential is valid.

Proposition 3.1 [3, 6] *Let $f : D_r \to \mathbb{R}$ be a locally bounded, integrable function in D_r and ω be the logarithmic potential of f. Then $\omega \in C^1(D_r)$ and for any $z = x_1 + ix_2 \in D_r$,*

$$\frac{\partial}{\partial x_j}\omega(z) = \frac{1}{2\pi} \iint_{D_r} \frac{\partial}{\partial x_j} \log|z - \zeta| f(\zeta) d\sigma_\zeta$$

for $j \in \{1, 2\}$.
If, in addition, f is locally Hölder continuous with exponent $\nu \leq 1$, then $\omega \in C^2(D_r)$, $\Delta\omega = f$ in D_r and for $z \in D_r$,

$$\begin{aligned} \frac{\partial^2}{\partial x_l \partial x_j}\omega(z) &= \frac{1}{2\pi} \iint_{D_R} \frac{\partial^2}{\partial x_l \partial x_j} \log|z - \zeta| \left(f(\zeta) - f(z)\right) d\sigma_\zeta \\ &\quad - \frac{1}{2\pi} f(z) \int_{\partial D_R} \frac{\partial}{\partial x_j} \log|z - \zeta| N_l(\zeta) |d\zeta|, \end{aligned}$$

where $N(\zeta) = (N_1(\zeta), N_2(\zeta))$ is the unit outward normal at the point $\zeta \in \partial D_R$, $R > r$ such that the divergence theorem holds on D_R and f is extended to vanish outside of D_r.

There is a similar proposition for higher order derivatives of the logarithmic potential. Define a multi-index $\mathbf{j} = (j_1, j_2)$, $|\mathbf{j}| = j_1 + j_2$, $j_1, j_2 = 0, 1, 2, \ldots$. For $z = x_1 + ix_2$, denote

$$\frac{\partial}{\partial x_1} = \partial_1, \ \frac{\partial}{\partial x_2} = \partial_2, \ \partial^{\mathbf{j}} = \partial_1^{j_1} \partial_2^{j_2} \text{ and } \frac{\partial^j}{\partial \zeta} = \frac{\partial^{j_1}}{\partial y_1} \frac{\partial^{j_2}}{\partial y_2}.$$

Let $e_\tau = (0,1)$ or $(1,0)$ for $\tau = 1, 2, \ldots$. Then $\mathbf{j}$ can be expressed in the form $e_1 + e_2 + \cdots + e_n$. We define two vectors $\theta_\tau := e_1 + \cdots + e_\tau$, $\phi_\tau := e_{\tau+2} + \cdots + e_n$ for $\tau = 1, \ldots, n-1$ where $\phi_{n-1} = (0,0)$, so $\mathbf{j}$ has a decomposition as $\mathbf{j} = \theta_\tau + e_{\tau+1} + \phi_\tau$. Write $\zeta = y_1 + iy_2$ and set

$$P_n[f](z, \zeta) := \begin{cases} \displaystyle\sum_{|\iota| \le n} \frac{(\zeta - z)^\iota \partial^\iota}{\iota!} f(z) & \text{if } \ n \ge 1 \\ f(z) & \text{if } \ n = 0, \end{cases}$$

where ι is a multi-index, $\iota = (\iota_1, \iota_2)$, $(\zeta - z)^\iota = (y_1 - x_1)^{\iota_1} (y_2 - x_2)^{\iota_2}$, $\iota! = \iota_1! \iota_2!$. We have the following recurrent formula for $P_n[f](z, \zeta)$.

Lemma 3.2 *For $P_n[f](z, \zeta)$ defined as above, then*

$$\frac{\partial^e}{\partial \zeta} P_n[f](z, \zeta) = P_{n-1}[\partial^e f](z, \zeta)$$

holds for $e = (0,1)$ or $(1,0)$.

Proof. We take the case $e = (1,0)$ as an example, when $e = (0,1)$ it is similar. Let $n \ge 1$. Then

$$\begin{aligned}
& \frac{\partial}{\partial y_1} P_n[f](z, \zeta) \\
= \ & \frac{\partial}{\partial y_1} \sum_{\substack{\iota_1 + \iota_2 \le n \\ 0 \le \iota_1 \le n}} \frac{(y_1 - x_1)^{\iota_1} (y_2 - x_2)^{\iota_2}}{\iota_1! \iota_2!} \partial_1{}^{\iota_1} \partial_2{}^{\iota_2} f(z) \\
= \ & \sum_{\substack{\iota_1 + \iota_2 \le n \\ 1 \le \iota_1 \le n}} \frac{(y_1 - x_1)^{\iota_1 - 1} (y_2 - x_2)^{\iota_2}}{(\iota_1 - 1)! \iota_2!} \partial_1{}^{\iota_1} \partial_2{}^{\iota_2} f(z) \\
= \ & \sum_{\substack{(\iota_1 - 1) + \iota_2 \le n-1 \\ 0 \le \iota_1 - 1 \le n}} \frac{(y_1 - x_1)^{\iota_1 - 1} (y_2 - x_2)^{\iota_2}}{(\iota_1 - 1)! \iota_2!} \partial_1{}^{\iota_1 - 1} \partial_2{}^{\iota_2} \, \partial_1 f(z) \\
= \ & \sum_{\iota_1 + \iota_2 \le n-1} \frac{(y_1 - x_1)^{\iota_1} (y_2 - x_2)^{\iota_2} \partial_1{}^{\iota_1} \partial_2{}^{\iota_2}}{\iota_1! \iota_2!} \partial_1 f(z) \\
= \ & \sum_{|\iota| \le n-1} \frac{(\zeta - z)^\iota \partial^\iota}{\iota!} \partial_1 f(z) \\
= \ & P_{n-1}[\partial_1 f](z, \zeta).
\end{aligned}$$

□

Using the multi-index notation, the Taylor expansion of f can be written in short.

Theorem 3.3 [cf. 1] *If $f(\zeta)$ is analytic in a domain $\Omega \in \mathbb{C}$, containing the point z, it is possible to write*

$$f(\zeta) = \sum_{t=0}^{n} \frac{f^{(t)}(z)}{t!}(\zeta - z) + R_{n+1}(z, \zeta),$$

where $R_{n+1}(z,\zeta)$ is the error term and $R_{n+1}(z,\zeta) = f_{n+1}(z)(\zeta - z)^{n+1}$ with $f_{n+1}(z)$ analytic in Ω. This expression is equivalent to

$$f(\zeta) = P_n[z](z, \zeta) + R_{n+1}(z, \zeta), \tag{3.2}$$

with $R_{n+1}(z,\zeta)$ as above.

Remark. If $f \in C^{n,\nu}(\Omega)$ with $0 < \nu \leq 1$, $n \geq 1$, and the Hölder continuity is a local property, then the error term $R_{n+1}(z,\zeta)$ satisfies

$$R_{n+1}(z, \zeta) = O(|z - \zeta|^{\nu+n}). \tag{3.3}$$

On the basis of Lemma 3.2, we can present the analogue of Proposition 3.1 as follows.

Proposition 3.4 *Let $r < 1$, $f : D_r \to \mathbb{R}$, $f \in C^{n-2,\nu}(D_r)$ with $0 < \nu \leq 1$, $n \geq 3$ and ω be the logarithmic potential of f. Then $\omega(z) \in C^n(D_r)$ and for a multi-index j, $|\boldsymbol{j}| = n$,*

$$\begin{aligned}\partial^{\boldsymbol{j}}\omega(z) \quad &= \quad \frac{1}{2\pi}\iint_{D_R} \partial^{\boldsymbol{j}} \log|z-\zeta| \cdot (f(\zeta) - P_{n-2}[f](z,\zeta))\, d\sigma_\zeta \\ &\quad -\frac{1}{2\pi}\sum_{\tau=1}^{n-1}\int_{\partial D_R} \partial^{\theta_\tau} \log|z-\zeta| \cdot P_{\tau-1}[\partial^{\phi_\tau} f](z,\zeta) \cdot \langle N(\zeta), e_{\tau+1}\rangle |d\zeta|, \end{aligned} \tag{3.4}$$

where $N(\zeta) = (N_1(\zeta), N_2(\zeta))$ is the unit outward normal at the point $\zeta \in \partial D_R$, $\langle\ ,\ \rangle$ is the inner product, $R > r$ such that the divergence theorem holds on D_R and the function f is extended to vanish outside of D_r.

We need the following Divergence Theorem for the proof. For a point $z = (x_1, x_2)$, a vector field $\mathbf{w}(z) = (w_1(z), w_2(z))$ and a function $u(z)$, denote

$$\operatorname{div} \mathbf{w} = \frac{\partial w_1}{\partial x_1} + \frac{\partial w_2}{\partial x_2} = \text{divergence of } \mathbf{w},$$

$$Du = (\partial_1 u, \partial_2 u) = \text{gradient of } u,$$

then $\Delta u = \operatorname{div} Du$.

Theorem 3.5 [cf. 3] (Divergence Theorem) *Let Ω be a bounded domain with C^1 boundary $\partial\Omega$, for any vector field $\mathbf{w}$ in $C^0(\bar{\Omega}) \cap C^1(\Omega)$, we have*

$$\iint_\Omega \operatorname{div} \mathbf{w} d\sigma_z = \int_{\partial\Omega} \langle N(z), \mathbf{w}\rangle\, d|z|, \tag{3.5}$$

where $\langle\ ,\ \rangle$ is the inner product.

In (3.5) we select $\mathbf{w}(z) = v(z)\,Du(z)$, then

$$\iint_\Omega Du\,Dv\,d\sigma_z + \iint_\Omega v\,Du\,d\sigma_z = \int_{\partial\Omega} v\,\langle Du, N(z)\rangle\,d|z|. \tag{3.6}$$

Since we only need one ∂_m for $m = 1, 2$, we can fix the the other component x_{3-m} in (3.6) and relabel u, we obtain the following *Green's (first) identity*:

$$\iint_\Omega u\,\partial_m v\,d\sigma_z + \iint_\Omega v\,\partial_m u\,d\sigma_z = \int_{\partial\Omega} uv\,N_m(z)\,d|z|. \tag{3.7}$$

Proof of Proposition 3.4. Let

$$\begin{aligned} u_j(z) \quad = \quad & \frac{1}{2\pi}\iint_{D_R} \partial^{\mathbf{j}} \log|z-\zeta|\cdot(f(\zeta) - P_{n-2}[f](z,\zeta))\,d\sigma_\zeta \\ & -\frac{1}{2\pi}\sum_{\tau=1}^{n-1}\int_{\partial D_R} \partial^{\theta_\tau}\log|z-\zeta|\cdot P_{\tau-1}[\partial^{\phi_\tau} f](z,\zeta)\cdot\langle N(\zeta), e_{\tau+1}\rangle|d\zeta|. \end{aligned} \tag{3.8}$$

Note that

$$\left|\partial^{\mathbf{j}}\log|z-\zeta|\right| \le \frac{n!}{|z-\zeta|^n}, \tag{3.9}$$

for $n = |j|$, and $\log|z-\zeta|$ is harmonic for $\zeta \neq z$, then by the local Hölder continuity of f in D_r, the function $u_j(z)$ is well defined.
Now we can employ induction. Since Proposition 3.4 has been obtained already, and j has the decomposition $j = \theta_{n-1} + e_n$, we may assume that the formula (3.4) is true for θ_{n-1}. Fix a function $\eta(t) \in C^{n-1}(\mathbb{R})$ such that $0 \le \eta \le 1$, $0 \le \eta^{(n-1)} \le 2$, $\eta(t) = 0$ for $t \le 1$, $\eta(t) = 1$ for $t \ge 2$, and set

$$\eta_\varepsilon := \eta(\frac{|z-\zeta|}{\varepsilon}),\ L := \frac{1}{2\pi}\log|z-\zeta|.$$

Note that η_ε and L are both skew symmetric with respect to x_1 and y_1, x_2 and y_2. Then

$$\partial^e L\eta_\varepsilon = -\frac{\partial^e}{\partial\zeta}L\eta_\varepsilon \tag{3.10}$$

for $e = (0, 1)$ or $(1, 0)$.
For $\varepsilon > 0$, define the function

$$v_j(z,\varepsilon) := \iint_{D_r} \partial^j L\eta_\varepsilon\cdot f(\zeta)d\sigma_\zeta.$$

We obtain $v_{\theta_{n-1}}(z,\varepsilon) \in C^{n-1}(D_r)$ for a fixed ε by induction.

From (3.9) we know that, $\zeta = z$ is a singularity of $\log|z-\zeta|$ when $|j| \ge 3$. To overwhelm the blow-up behavior near the singularity, we need the Taylor expansion (3.2). To prevent a singularity from appearing on the boundary $\partial\Omega$, we have to enlarge the domain D_r of the integral (3.4) into a larger domain D_R where the divergence theorem holds. Thus for sufficiently small ε,

$$\begin{aligned} & \partial^{e_n} v_j(z,\varepsilon) = \iint_{D_r} \partial^{e_n}(\partial^{\theta_{n-1}}L\eta_\varepsilon)\cdot f(\zeta)d\sigma_\zeta \\ = \quad & \iint_{D_R} \partial^{e_n}(\partial^{\theta_{n-1}}L\eta_\varepsilon)\cdot(f(\zeta) - P_{n-2}[f])\,d\sigma_\zeta + \iint_{D_R} \partial^{e_n}(\partial^{\theta_{n-1}}L\eta_\varepsilon)\cdot P_{n-2}[f]d\sigma_\zeta. \end{aligned}$$

Combining the skew symmetry (3.10), Green's identity (3.7) and Theorem 3.2, for sufficiently small ε, we have

$$
\begin{aligned}
& \iint_{D_R} \partial^{e_n}(\partial^{\theta_{n-1}} L\eta_\varepsilon) \cdot P_{n-2}[f] d\sigma_\zeta \\
= & -\iint_{D_R} \frac{\partial^{e_n}}{\partial \zeta} \partial^{e_n}(\partial^{\theta_{n-1}} L\eta_\varepsilon) \cdot P_{n-2}[f] d\sigma_\zeta \\
= & -\int_{\partial D_R} \partial^{\theta_{n-1}} L \cdot P_{n-2}[f] \langle N(\zeta), e_n\rangle |d\zeta| + \iint_{D_R} \partial^{\theta_{n-1}} L\eta_\varepsilon \cdot P_{n-3}[\partial^{e_n} f] d\sigma_\zeta \\
= & \ \ldots \\
= & -\int_{\partial D_R} \partial^{\theta_{n-1}} L \cdot P_{n-2}[f] \langle N(\zeta), e_n\rangle |d\zeta| - \ldots - \int_{\partial D_R} \partial^{\theta_2} L \cdot P_1[\partial^{\phi_2} f] \langle N(\zeta), e_3\rangle |d\zeta| \\
& + \iint_{D_R} \partial^{\theta_2} L\eta_\varepsilon \cdot P_0[\partial^{\phi_1} f] d\sigma_\zeta \\
= & -\int_{\partial D_R} \partial^{\theta_{n-1}} L \cdot P_{n-2}[f] \langle N(\zeta), e_n\rangle |d\zeta| - \ldots - \int_{\partial D_R} \partial^{\theta_2} L \cdot P_1[\partial^{\phi_2} f] \langle N(\zeta), e_3\rangle |d\zeta| \\
& - \int_{\partial D_R} \partial^{\theta_1} L \cdot P_0[\partial^{\phi_1} f] \langle N(\zeta), e_2\rangle |d\zeta| \\
= & -\sum_{\tau=1}^{n-1} \int_{\partial D_R} \partial^{\theta_\tau} L \cdot P_{\tau-1}[\partial^{\phi_\tau} f] \langle N(\zeta), e_{\tau+1}\rangle |d\zeta|.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\partial^{e_n} v_j(z,\varepsilon) = & \iint_{D_R} \partial^{e_n}(\partial^{\theta_{n-1}} L\eta_\varepsilon) \cdot (f(\zeta) - P_{n-2}[f])\, d\sigma_\zeta \\
& - \sum_{\tau=1}^{n-1} \int_{\partial D_R} \partial^{\theta_\tau} L \cdot P_{\tau-1}[\partial^{\phi_\tau} f] \langle N(\zeta), e_{\tau+1}\rangle |d\zeta|.
\end{aligned} \tag{3.11}
$$

Now we compare (3.8) and (3.11). By the local Hölder continuity of f, Theorem 3.3 and the estimate (3.3), there exist constants M_1 and M_2 such that

$$
\begin{aligned}
& |u_j(z) - \partial^{e_n} v_j(z,\varepsilon)| \\
= & \left| \iint_{|\zeta - z| \le 2\varepsilon} \left(\partial^j L - \partial^j L\eta_\varepsilon\right) R_{n-1}(z,\zeta) d\sigma_\zeta \right| \\
\le & \ M_1 \iint_{|\zeta - z| \le 2\varepsilon} \left(\frac{n!}{|\zeta - z|^n} + \frac{2(n-1)!}{\varepsilon |\zeta - z|^{n-1}} \right) |z - \zeta|^{\nu + n - 2} d\sigma_\zeta \\
= & \ M_1 \iint_{|\zeta - z| \le 2\varepsilon} \left(\frac{n!}{|\zeta - z|^2} + \frac{2(n-1)!}{\varepsilon |\zeta - z|} \right) |z - \zeta|^{\nu} d\sigma_\zeta \\
\le & \ M_2 \cdot (2\varepsilon)^\nu
\end{aligned}
$$

The last inequality comes from Lemma 4.2 in [3]. Hence $\partial^{e_n} v_j(z,\zeta)$ converges to $u_j(z)$ uniformly on any compact subset of D_r as $\varepsilon \to 0$. It is easy to see $v_j(z,\varepsilon)$ converges uniformly to $\partial^{\theta_{n-1}}\omega$ in the disk D_r, then $\omega \in C^n(D_r)$ and $u_j(z) = \partial^j \omega(z)$. The proof is complete. □

We list two results on a class of conformal metrics with negative curvatures as an application of potential theory. No proof is involved here. For more details, see [6, 7], also [11].

Theorem 3.6 [6] *Let $\kappa : \mathbb{D} \to \mathbb{R}$ be a locally Hölder continuous function with $\kappa(0) < 0$. If $u : \mathbb{D}^* \to \mathbb{R}$ is a C^2-solution to $\Delta u = -\kappa(z)e^{2u}$ in $\mathbb{D}^*$, then u has the order $\alpha \in (-\infty, 1]$ and*

$$\begin{aligned} u(z) &= -\alpha \log|z| + v(z), && \text{if } \alpha < 1, \\ u(z) &= -\log|z| - \log\log(1/|z|) + w(z), && \text{if } \alpha = 1, \end{aligned}$$

where the remainder functions $v(z)$ and $w(z)$ are continuous in $\mathbb{D}$. Moreover, the second partial derivatives satisfy the following,

$$\begin{aligned} & v_{zz}(z), v_{z\bar{z}}(z) \text{ and } v_{\bar{z}\bar{z}}(z) \text{ are continuous at } z = 0 && \text{if } \alpha \leq 0; \\ & v_{zz}(z), v_{z\bar{z}}(z), v_{\bar{z}\bar{z}}(z) = O(|z|^{-2\alpha}) && \text{if } 0 < \alpha < 1, \\ & w_{zz}(z), w_{\bar{z}\bar{z}}(z), w_{z\bar{z}}(z) = O(|z|^{-2}\log^{-2}(1/|z|)) && \text{if } \alpha = 1, \end{aligned}$$

when z tends to $z = 0$.

Theorem 3.7 [11] *Let $\kappa : \mathbb{D} \to \mathbb{R}$ satisfy $\kappa(0) < 0$, $\kappa(z) \in C^{n-2,\nu}(\mathbb{D}^*)$ for an integer $n \geq 3$, $0 < \nu \leq 1$ and let $u : \mathbb{D}^* \to \mathbb{R}$ be a $C^{n,\nu}$-solution to $\Delta u = -\kappa(z)e^{2u}$ in $\mathbb{D}^*$. Then $u(z)$ has a singularity at the origin of the order $0 < \alpha \leq 1$, and for $n_1,\ n_2 \geq 1$, $n_1 + n_2 = n$, near the origin, $v(z)$, $w(z)$ as in Theorem 3.6 satisfy*

$$\partial^n v(z),\ \bar{\partial}^n v(z),\ \bar{\partial}^{n_1}\partial^{n_2} v(z) = O(|z|^{2-2\alpha-n}),$$

$$\bar{\partial}^n w(z),\ \partial^n w(z) = O(|z|^{-n}\log^{-2}(1/|z|)),$$
$$\bar{\partial}^{n_1}\partial^{n_2} w(z) = O(|z|^{-n}\log^{-3}(1/|z|)),$$

where

$$\partial^n = \frac{\partial^n}{\partial z^n},\ \bar{\partial}^n = \frac{\partial^n}{\partial \bar{z}^n}$$

for a positive natural number n.

Acknowledgement. The author would like to thank Prof. Toshiyoki Sugawa and Prof. Le Hung Son for their helpful comments, suggestion and encouragement.

References

[1] L. Ahlfors, *An extension of Schwarz's lemma*, Trans. Amer. Math. Soc. **42** (1938), 359–364.

[2] ———, *Complex analysis*, McGraw, 1973.

[3] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Springer, Berlin–New York, 1997.

[4] M. Heins, *On a class of conformal metrics*, Nagoya Math. J. **21** (1962), 1–60.

[5] D. Kraus and O. Roth, *On the isolated singularities of the solutions of the Gaussian curvature equation for nonnegative curvature*, J. Math. Anal. Appl. **345** (2008), no. 2, 628–631.

[6] ——— , *The behaviour of solutions of the Gaussian curvature equation near an isolated boundary point*, Math. Proc. Cambridge Phil. Soc. **145** (2008), 643–667.

[7] D. Kraus, O. Roth and T. Sugawa, *Metrics with conical singularities on the sphere and sharp extensions of the theorems of Landau and Schottky*, Math. Z. **267** (2011), 851–868.

[8] R. C. McOwen, *Point singularities and conformal metrics on riemann surfaces*, Proc. Amer. Math. Soc. **103** (1988), no. 1, 222–224.

[9] J. Nitsche, *Über die isolierten Singularitäten der Lösungen von $\Delta u = e^u$*, Math. Z. **68** (1957), 316–324.

[10] M. Troyanov, *Prescribing curvature on compact surfaces with conical singularities*, Trans. Amer. Math. Soc. **324** (1990), no. 2, 793–821.

[11] T. Zhang, *A note on the asymptotic behavior of conformal metrics with negative curvatures near isolated singularities*, (in preparation).